MODIFIED-LIKELIHOOD ESTIMATION OF FIXED-EFFECT MODELS FOR DYADIC DATA

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We consider point estimation and inference based on modifications of the profile likelihood in models for dyadic interactions between n agents featuring agent-specific parameters. This setup covers the β-model of network formation and generalizations thereof. The maximum-likelihood estimator of such models has bias and standard deviation of $O(n^{-1})$ and so is asymptotically biased. Estimation based on modified likelihoods leads to estimators that are asymptotically unbiased and likelihood-ratio tests that exhibit correct size. We apply the modifications to versions of the β-model for network formation and of the Bradley-Terry model for paired comparisons.

1. Introduction. A growing literature has uncovered the importance of interactions between agents through networks as drivers for economic and social outcomes. A leading approach to statistical modelling of dyadic interaction is through the inclusion of agent-specific parameters (see, e.g., Snijders 2011 for many references). A specific example that has received substantial attention in the recent literature is the β-model for network formation. There, agent fixed effects serve to capture degree heterogeneity in link formation and the inclusion of dyad-level covariates reflects homophily. Recent theoretical work on the β-model includes Chatterjee, Diaconis and Sly [2011], Rinaldo, Petrovic and Fienberg [2013], Yan and Xu [2013], and Graham [2017].

Estimation of fixed-effect models for dyadic data is non-standard as the number of parameters grows with the sample size, and inference on common parameters is plagued by asymptotic bias that needs to be corrected for.

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The bias problem comes from the presence of the agent-specific parameters in the model, and is similar to the well-known incidental-parameter problem (Neyman and Scott 1948) in models for panel data. Graham [2017] derives the leading bias of the maximum-likelihood estimator in the β-model and considers correcting for it.

The problem of inference in the presence of many nuisance parameters has a long history. In this paper we look at generic estimation problems for dyadic data and argue in favor of inference based on modifying the likelihood function. In its most general form, the modified likelihood is a bias-corrected version of the profile likelihood, that is, of the likelihood after having profiled-out the nuisance parameters. The adjustment is both general and simple in form, involving only the score and Hessian of the likelihood with respect to the nuisance parameters. The adjustment term removes the leading bias from the profile likelihood and leads to asymptotically-unbiased inference and likelihood ratio statistics that are χ^2-distributed under the null. The form of the adjustment can be specialized by using the likelihood structure (as in DiCiccio et al. 1996), in which case the modified likelihood penalizes the profile likelihood for deviations from the information equality, arising due to the estimation noise in the fixed effects.[1]

We work out the modifications to the profile likelihood in a linear version of the β-model and in a linear version of the Bradley and Terry [1952] model for paired comparisons. These simple illustrations give insight in how the adjustments work. We next apply them to the β-model of Graham [2017], and evaluate our approach using his simulation designs. We find that both modifications dramatically improve on maximum likelihood in terms of bias and mean squared error as well as reliability of statistical inference, and that they are considerably more reliable than ex-post bias-correction of the maximum-likelihood estimator.

2. Fixed-effect models for dyadic data.

We consider data on dyadic interactions between n agents. For each of $n(n-1)/2$ distinct agent pairs (i, j) with $i < j$ we observe the random variable z_{ij} (which may be vector

[1] It can be further simplified when an information-orthogonal reparametrization exist, as in Cox and Reid [1987] and Lancaster [2002]. However, as such reparametrizations do not exist in general (see, e.g., Severini 2000) we do not consider such modifications further here.

valued). For example, we may observe an outcome y_{ij} generated by pair (i, j) together with a vector of dyad characteristics x_{ij}, in which case we have $z_{ij} = (y_{ij}, x'_{ij})'$.

The density of z_{ij} (relative to some dominating measure) takes the form

$$f(z_{ij}; \vartheta, \beta_i, \beta_j),$$

where ϑ and $\beta_1, \ldots, \beta_n$ are unknown Euclidean parameters. Models of this form are relevant in many areas. Examples include the analysis of network formation (Holland and Leinhardt 1981), the study of strategic behavior among agents (Bajari, Hong and Nekipelov 2010) as well as matching and sorting in bipartite networks (Abowd, Kramarz and Margolis 1999), and the construction of rankings (Bradley and Terry 1952).

Our goal will be to perform inference on ϑ treating the β_i as fixed effects. As is well known, the maximum-likelihood estimator of ϑ generally performs poorly when the number of nuisance parameters is large relative to the sample size (Neyman and Scott 1948). We will consider modifications of the maximum-likelihood method that yield asymptotically-unbiased estimators that achieve the Cramér-Rao bound and likelihood-ratio statistics that yield consistent tests that are size-correct in large samples.

3. Estimation and inference. The log-likelihood is

$$\ell(\vartheta, \beta) = \sum_{i=1}^{n} \sum_{i<j} \log f(z_{ij}; \vartheta, \beta_i, \beta_j),$$

where we let $\beta = (\beta_1, \ldots, \beta_n)'$. For simplicity of exposition we ignore any normalization that may be needed on β to achieve identification. When a normalization of the form $c(\beta) = 0$ is needed, everything to follow goes through on replacing $\ell(\vartheta, \beta)$ by the constrained likelihood $\ell(\vartheta, \beta) - \lambda\, c(\beta)$, where λ denotes the Lagrange multiplier. We will give a detailed example below.

It is useful to recall that the maximum-likelihood estimator of ϑ can be expressed as

$$\hat{\vartheta} = \arg\max_{\vartheta} \hat{\ell}(\vartheta),$$

where $\hat{\ell}(\vartheta) = \ell(\vartheta, \hat{\beta}(\vartheta))$, with

$$\hat{\beta}(\vartheta) = \arg\max_{\beta} \ell(\vartheta, \beta),$$

is the profile likelihood.

Inference based on the profile likelihood performs poorly, even in large samples, because the dimension of β is n, which grows with the sample size $n(n-1)/2$. Quite generally, estimating the n parameters β_i along with ϑ will imply that

$$\mathbb{E}(\hat{\vartheta} - \vartheta) = O(n^{-1}).$$

As $\mathbb{E}((\hat{\vartheta} - \vartheta)^2) = O(n^{-2})$, bias and standard deviation are of the same order of magnitude, and the maximum-likelihood estimator is asymptotically biased.

3.1. *Modified profile likelihood.* Estimation and inference in the presence of (many) nuisance parameters has a long history. Seminal contributions of Barndorff-Nielsen [1983] and Cox and Reid [1987] contains modifications to the profile likelihood that lead to superior inference. More recent work includes DiCiccio et al. [1996] and Severini [1998]. Modified likelihoods have been found to solve the incidental-parameter problem in models for panel data under so-called rectangular-array asymptotics (as defined in Li, Lindsay and Waterman 2003); see Sartori [2003]. Here we wish to argue that they can equally be used to yield asymptotically-valid inference in the current context.

In its simplest form, modified likelihoods can be understood as yielding a superior approximation to the target likelihood

$$\ell(\vartheta) = \ell(\vartheta, \beta(\vartheta)), \qquad \beta(\vartheta) = \arg\max_{\beta} \mathbb{E}(\ell(\vartheta, \beta)).$$

Moreover, the profile likelihood is the sample counterpart to this infeasible likelihood. Replacing $\beta(\vartheta)$ with $\hat{\beta}(\vartheta)$ introduces bias that leads to invalid inference.

Under regularity conditions similar to those in, say Sartori [2003], we have

$$\hat{\beta}(\vartheta) - \beta(\vartheta) = \varSigma(\vartheta)^{-1} V(\vartheta) + O_p(n^{-1}),$$

where we introduce

$$V(\vartheta) = \left. \frac{\partial \ell(\vartheta, \beta)}{\partial \beta} \right|_{\beta = \beta(\vartheta)}, \qquad \varSigma(\vartheta) = - \left. \mathbb{E}\left(\frac{\partial^2 \ell(\vartheta, \beta)}{\partial \beta \partial \beta'} \right) \right|_{\beta = \beta(\vartheta)}.$$

An expansion of the profile likelihood around $\beta(\vartheta)$ yields

$$\begin{aligned}\hat{\ell}(\vartheta)-\ell(\vartheta) = \quad & (\hat{\beta}(\vartheta)-\beta(\vartheta))'V(\vartheta) \\ & -\frac{1}{2}(\hat{\beta}(\vartheta)-\beta(\vartheta))'\Sigma(\vartheta)(\hat{\beta}(\vartheta)-\beta(\vartheta)) + O_p(n^{-1/2}).\end{aligned}$$

Combining the two expansions and taking expectations then shows that the bias of the profile likelihood is of the form

$$\mathbb{E}(\hat{\ell}(\vartheta)-\ell(\vartheta)) = \frac{1}{2}\mathrm{trace}(\Sigma(\vartheta)^{-1}\Omega(\vartheta)) + O(n^{-1/2})$$

for

$$\Omega(\vartheta) = \mathbb{E}[V(\vartheta)\,V(\vartheta)'],$$

the variance of $V(\vartheta)$.

A modified likelihood then is

$$\dot{\ell}(\vartheta) = \hat{\ell}(\vartheta) - \frac{1}{2}\mathrm{trace}(\hat{\Sigma}(\vartheta)^{-1}\hat{\Omega}(\vartheta)),$$

where we define the plug-in estimators

$$\hat{\Sigma}(\vartheta) = \hat{\Sigma}(\vartheta,\hat{\beta}(\vartheta)), \qquad \hat{\Omega}(\vartheta) = \hat{\Omega}(\vartheta,\hat{\beta}(\vartheta)),$$

for matrices

$$-(\hat{\Sigma}(\vartheta,\beta))_{i,j} = \begin{cases} \sum_{i<k} \frac{\partial^2 \log f(z_{ik};\vartheta,\beta_i,\beta_k)}{\partial\beta_i^2} + \sum_{i>k} \frac{\partial^2 \log f(z_{ki};\vartheta,\beta_k,\beta_i)}{\partial\beta_i^2} & \text{if } i=j \\ \frac{\partial^2 \log f(z_{ij};\vartheta,\beta_i,\beta_j)}{\partial\beta_i\partial\beta_j} & \text{if } i<j \\ \frac{\partial^2 \log f(z_{ji};\vartheta,\beta_j,\beta_i)}{\partial\beta_i\partial\beta_j} & \text{if } i>j \end{cases}$$

and

$$(\hat{\Omega}(\vartheta,\beta))_{i,j} = \begin{cases} \sum_{i<k} \left(\frac{\partial \log f(z_{ik};\vartheta,\beta_i,\beta_k)}{\partial\beta_i}\right)^2 + \sum_{i>k} \left(\frac{\partial \log f(z_{ki};\vartheta,\beta_k,\beta_i)}{\partial\beta_i}\right)^2 & \text{if } i=j \\ \left(\frac{\partial \log f(z_{ij};\vartheta,\beta_i,\beta_j)}{\partial\beta_i}\right)^2 & \text{if } i<j \\ \left(\frac{\partial \log f(z_{ji};\vartheta,\beta_j,\beta_i)}{\partial\beta_i}\right)^2 & \text{if } i>j \end{cases}$$

In large samples, this modification removes the leading bias from the profile likelihood. Consequently, in large samples, the likelihood-ratio statistic has correct size and

$$\dot{\vartheta} = \arg\max_{\vartheta} \dot{\ell}(\vartheta),$$

will have bias $o(n^{-1})$. Furthermore, under usual regularity conditions, we have the limit result

$$(\dot{\vartheta} - \vartheta) \overset{a}{\sim} N\left(0, \frac{I(\vartheta)^{-1}}{n(n-1)/2}\right)$$

as $n \to \infty$, where we let

$$I(\vartheta) = \lim_{n\to\infty} \mathbb{E}\left(-\frac{\partial^2 \ell(\vartheta)}{\partial\vartheta\partial\vartheta'}\right) \Big/ \left(\frac{n(n-1)}{2}\right)$$

be the Fisher information for ϑ.

The only point at which the likelihood setting has been used so far is in the statement of the limit distribution of $\dot{\vartheta} - \vartheta$, where the expression for the asymptotic variance exploits the information equality. Bias-corrected estimation—using the same formula for the bias as before—thus carries over to more general extremum-type estimation problems; the only change being that, now, the asymptotic variance is $I(\vartheta)^{-1} \Omega(\vartheta)\, I(\vartheta)^{-1}$.

Alternatively, following the arguments in Arellano and Hahn [2007] we can exploit the likelihood structure to get

$$\frac{1}{2}\mathrm{trace}(\hat{\Sigma}(\vartheta)^{-1}\hat{\Omega}(\vartheta)) = -\frac{1}{2}\log(\det \hat{\Sigma}(\vartheta)) + \frac{1}{2}\log(\det \hat{\Omega}(\vartheta)) + O(n^{-1}),$$

which validates the alternative modified likelihood

$$\ddot{\ell}(\vartheta) = \hat{\ell}(\vartheta) + \frac{1}{2}\log(\det \hat{\Sigma}(\vartheta)) - \frac{1}{2}\log(\det \hat{\Omega}(\vartheta));$$

see DiCiccio et al. [1996]. Its maximizer, say $\ddot{\vartheta}$, satisfies the same asymptotic properties as $\dot{\vartheta}$.

3.2. *Illustration: A linear β-model.* Consider the following extension of the classic many normal means problem of Neyman and Scott [1948]. Data are generated as

$$z_{ij} \sim N(\beta_i + \beta_j, \vartheta),$$

and are independent across dyads. The likelihood function for all parameters (ignoring constants) is

$$\ell(\vartheta, \beta) = -\frac{1}{2}\frac{n(n-1)}{2}\log\vartheta - \frac{1}{2}\sum_{i=1}^{n}\sum_{i<j}\frac{(z_{ij} - \beta_i - \beta_j)^2}{\vartheta}.$$

Its first two derivatives with respect to the β_i are

$$\frac{\partial \ell(\vartheta,\beta)}{\partial \beta_i} = \sum_{i<j} \frac{z_{ij}-\beta_i-\beta_j}{\vartheta} + \sum_{i>j} \frac{z_{ji}-\beta_j-\beta_i}{\vartheta}$$

and

$$\frac{\partial^2 \ell(\vartheta,\beta)}{\partial \beta_i \partial \beta_j} = \begin{cases} -\frac{(n-1)}{\vartheta} & \text{if } i=j \\ -\frac{1}{\vartheta} & \text{if } i \neq j \end{cases}.$$

Let $\tilde{z}_i = (n-2)^{-1}\sum_{i<j} z_{ij} + (n-2)^{-1}\sum_{i>j} z_{ji}$ and $\overline{z} = (2(n-1)^{-1}\sum_{i=1}^n \tilde{z}_i$. Solving for the maximum-likelihood estimator of β_i gives $\hat{\beta}_i = \tilde{z}_i - \overline{z}$ for any ϑ. The profile likelihood is therefore

$$\hat{\ell}(\vartheta) = -\frac{n(n-1)}{2}\log\vartheta - \frac{1}{2}\sum_{i=1}^n \sum_{i<j} \frac{(z_{ij}-(\tilde{z}_i-\overline{z})-(\tilde{z}_j-\overline{z}))^2}{\vartheta},$$

and its maximizer is

$$\hat{\vartheta} = \frac{2}{n(n-1)}\sum_{i=1}^n \sum_{i<j} (z_{ij}-(\tilde{z}_i-\overline{z})-(\tilde{z}_j-\overline{z}))^2.$$

Some tedious but straightforward calculations yield

$$\mathbb{E}(\hat{\vartheta}-\vartheta) = -\frac{2}{n-1}\,\vartheta, \qquad \operatorname{var}(\hat{\vartheta}) = \frac{n-3}{n-1}\,\frac{2\,\vartheta^2}{n(n-1)/2},$$

which confirms that the maximum-likelihood estimator of ϑ suffers from asymptotic bias. Moreover,

$$\sqrt{\frac{n(n-1)}{2}}(\hat{\vartheta}-\vartheta) \xrightarrow{d} N\big(-\sqrt{2}\vartheta, (\sqrt{2}\vartheta)^2\big),$$

as $n \to \infty$.

To set up the modified likelihood, first note that

$$(\hat{\Sigma}(\vartheta))_{i,j} = \begin{cases} \frac{n-1}{\vartheta} & \text{if } i=j \\ \frac{1}{\vartheta} & \text{if } i \neq j \end{cases}, \quad (\hat{\Sigma}(\vartheta)^{-1})_{i,j} = \begin{cases} \frac{\vartheta}{2}\frac{2n-3}{(n-1)(n-2)} & \text{if } i=j \\ -\frac{\vartheta}{2}\frac{1}{(n-1)(n-2)} & \text{if } i \neq j \end{cases},$$

and that

$$(\hat{\Omega}(\vartheta))_{i,j} = \begin{cases} \sum_{i<k} \frac{(z_{ik}-(\tilde{z}_i-\overline{z})-(\tilde{z}_k-\overline{z}))^2}{\vartheta^2} + \sum_{i>k} \frac{(z_{ki}-(\tilde{z}_k-\overline{z})-(\tilde{z}_i-\overline{z}))^2}{\vartheta^2} & \text{if } i=j \\ \frac{(z_{ij}-(\tilde{z}_i-\overline{z})-(\tilde{z}_j-\overline{z}))^2}{\vartheta^2} & \text{if } i<j \\ \frac{(z_{ji}-(\tilde{z}_j-\overline{z})-(\tilde{z}_i-\overline{z}))^2}{\vartheta^2} & \text{if } i>j \end{cases}.$$

It is then easily seen that

$$\frac{1}{2}\mathrm{trace}(\hat{\Sigma}(\vartheta)^{-1}\hat{\Omega}(\vartheta)) = \frac{1}{2}\,\frac{2}{n-1}\sum_{i=1}^{n}\sum_{i<j}\frac{(z_{ij}-(\tilde{z}_i-\overline{z})-(\tilde{z}_j-\overline{z}))^2}{\vartheta}.$$

From this we obtain

$$\dot{\ell}(\vartheta) = -\frac{n(n-1)}{2}\log\vartheta - \left(1+\frac{2}{n-1}\right)\frac{1}{2}\sum_{i=1}^{n}\sum_{i<j}\frac{(z_{ij}-(\tilde{z}_i-\overline{z})-(\tilde{z}_j-\overline{z}))^2}{\vartheta},$$

and its maximizer

$$\dot{\vartheta} = \frac{n+1}{n-1}\,\hat{\vartheta} = \hat{\vartheta} + \frac{2}{n-1}\hat{\vartheta}.$$

Clearly, this estimator removes the leading bias from the maximum-likelihood estimator. Moreover,

$$\mathbb{E}(\dot{\vartheta}-\vartheta) = -\left(\frac{2}{n-1}\right)^2\vartheta, \qquad \mathrm{var}(\dot{\vartheta}) = \frac{n(n(n-1)-5)}{(n-1)^3}\,\frac{2\,\vartheta^2}{n(n-1)/2},$$

which shows that the remaining bias in the point estimator is small relative to its standard deviation.

As an alternative correction, we may exploit the likelihood structure to adjust the profile likelihood by the term

$$-\frac{1}{2}\log(\det\hat{\Sigma}(\vartheta)) + \frac{1}{2}\log(\det\hat{\Omega}(\vartheta)) = \frac{n}{2}\,\log\,\vartheta + c,$$

where c is a constant that does not depend on ϑ. This yields the modification

$$\ddot{\ell}(\vartheta) = -\frac{n(n-3)}{2}\log\vartheta - \frac{1}{2}\sum_{i=1}^{n}\sum_{i<j}\frac{(z_{ij}-(\tilde{z}_i-\overline{z})-(\tilde{z}_j-\overline{z}))^2}{\vartheta},$$

whose maximizer satisfies

$$\mathbb{E}(\ddot{\vartheta}-\vartheta) = 0, \qquad \mathrm{var}(\ddot{\vartheta}) = \frac{2\,\vartheta^2}{n(n-3)/2}.$$

This estimator is exactly unbiased.

To give an idea of the magnitude of the bias in this problem, Table 1 contains the bias and standard deviation of the estimators $\hat{\vartheta}$, $\dot{\vartheta}$, and $\ddot{\vartheta}$ for various sample sizes n and variance parameter fixed to $\vartheta = 1$. These results are invariant to the value of the β_i and can be interpreted as relative bias for general values of ϑ.

TABLE 1
Many normal means

n	$\hat{\vartheta}$	$\dot{\vartheta}$	$\ddot{\vartheta}$	$\hat{\vartheta}$	$\dot{\vartheta}$	$\ddot{\vartheta}$
		bias			standard deviation	
5	-0.5000	-0.2500	0.0000	0.3162	0.4743	0.6325
10	-0.2222	-0.0494	0.0000	0.1859	0.2272	0.2390
15	-0.1429	-0.0204	0.0000	0.1278	0.1460	0.1491
20	-0.1053	-0.0111	0.0000	0.0970	0.1073	0.1085
25	-0.0833	-0.0069	0.0000	0.0782	0.0847	0.0853
50	-0.0408	-0.0017	0.0000	0.0396	0.0412	0.0413
75	-0.0270	-0.0007	0.0000	0.0265	0.0272	0.0272
100	-0.0202	-0.0004	0.0000	0.0199	0.0203	0.0203

3.3. *Illustration: A linear Bradley-Terry model.* As an alternative to the Neyman and Scott [1948] model with complementarities, now suppose that

$$z_{ij} \sim N(\beta_i - \beta_j, \vartheta)$$

independently across dyads. This model is overparametrized as, clearly, the mean of the β_i is not identified. A common normalization in this type of model is $\sum_{i=1}^n \beta_i = 0$ (Simons and Yao 1999), and we will maintain it here. The constrained likelihood is

$$-\frac{1}{2}\frac{n(n-1)}{2}\log\vartheta - \frac{1}{2}\sum_{i=1}^n\sum_{i<j}\frac{(z_{ij}-\beta_i+\beta_j)^2}{\vartheta} + \lambda\sum_{i=1}^n \beta_i,$$

where λ is the Lagrange multiplier for our normalization constraint. The first-order condition for the constrained problem for β_i for a given ϑ equals

$$\frac{\sum_{i<j} z_{ij} - \sum_{i>j} z_{ji}}{\vartheta} - \frac{n}{\vartheta}\beta_i = 0.$$

This gives

$$\hat{\beta}_i = \frac{\sum_{i<j} z_{ij} - \sum_{i>j} z_{ji}}{n} = \tilde{z}_i \quad \text{(say)}$$

for all i and any ϑ. Observe that the sign of $\hat{\beta}_i$ is driven by the comparison of the magnitudes of $\sum_{i<j} z_{ij}$ and $\sum_{i>j} z_{ji}$. Also note that $\sum_{i=1}^n \hat{\beta}_i = 0$ holds. We therefore have

$$\hat{\ell}(\vartheta) = -\frac{1}{2}\frac{n(n-1)}{2}\log\vartheta - \frac{1}{2}\sum_{i=1}^n\sum_{i<j}\frac{(z_{ij}-\tilde{z}_i+\tilde{z}_j)^2}{\vartheta},$$

and with it, the maximum-likelihood estimator

$$\hat{\vartheta} = \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{i<j} (z_{ij} - \tilde{z}_i + \tilde{z}_j)^2.$$

A calculation shows that $\mathbb{E}(\hat{\vartheta} - \vartheta) = -2n^{-1}\vartheta$.

It is immediate that

$$\hat{\Sigma}(\vartheta) = \operatorname{diag}\left(\frac{n}{\vartheta}\right), \qquad \hat{\Sigma}(\vartheta)^{-1} = \operatorname{diag}\left(\frac{\vartheta}{n}\right),$$

and that

$$(\hat{\Omega}(\vartheta))_{i,j} = \begin{cases} \sum_{i<k} \frac{(z_{ik}-\tilde{z}_i+\tilde{z}_k)^2}{\vartheta^2} + \sum_{i>k} \frac{(z_{ki}-\tilde{z}_k+\tilde{z}_i)^2}{\vartheta^2} & \text{if } i = j \\ \frac{(z_{ij}-\tilde{z}_i+\tilde{z}_j)^2}{\vartheta^2} & \text{if } i < j \\ \frac{(z_{ji}-\tilde{z}_j+\tilde{z}_i)^2}{\vartheta^2} & \text{if } i > j \end{cases}.$$

Therefore,

$$\dot{\ell}(\vartheta) = -\frac{1}{2}\frac{n(n-1)}{2}\log\vartheta - \frac{1}{2}\left(1+\frac{2}{n}\right)\sum_{i=1}^{n}\sum_{i<j}\frac{(z_{ij}-\tilde{z}_i+\tilde{z}_j)^2}{\vartheta},$$

$$\ddot{\ell}(\vartheta) = -\frac{1}{2}\frac{n(n-3)}{2}\log\vartheta - \frac{1}{2}\sum_{i=1}^{n}\sum_{i<j}\frac{(z_{ij}-\tilde{z}_i+\tilde{z}_j)^2}{\vartheta}.$$

The corresponding estimators are

$$\dot{\vartheta} = \left(1+\frac{2}{n}\right)\hat{\vartheta}, \qquad \ddot{\vartheta} = \frac{n-1}{n-3}\,\hat{\vartheta} = \left(1+\frac{2}{n-3}\right)\hat{\vartheta}.$$

Both remove the leading bias from the maximum-likelihood estimator, as

$$\mathbb{E}(\dot{\vartheta}-\vartheta) = -\frac{4}{n^2}\vartheta = O(n^{-2}), \qquad \mathbb{E}(\ddot{\vartheta}-\vartheta) = \frac{2}{n(n-3)}\vartheta = O(n^{-2}),$$

but, in this case, neither is exactly unbiased. The first estimator has bias that is strictly negative (for any finite n). The second estimator overcorrects and has strictly positive bias. The second-order bias is monotone in n. We have

$$\frac{4}{n^2}\vartheta > \frac{2}{n(n-3)}\vartheta$$

for all $n > 7$. As $n \to \infty$,

$$\sqrt{\frac{n(n-1)}{2}}(\dot{\vartheta} - \vartheta) \xrightarrow{d} N(0, 2\vartheta^2),$$

and $\|\ddot{\vartheta} - \dot{\vartheta}\| = o_p(n^{-1})$; that is, the two modifications to the likelihood yield asymptotically-equivalent estimators.

4. Application to the $\boldsymbol{\beta}$-model. The β-model of network formation (Chatterjee, Diaconis and Sly 2011; Graham 2017) models Bernoulli outcome variables as having success probability

$$\mathbb{P}(y_{ij} = 1 | x_{ij}; \vartheta, \beta_i, \beta_j) = F(\beta_i + \beta_j + x_{ij}'\vartheta),$$

where $F(a) = (1+e^{-a})^{-1}$ is the logit link function. Graham [2017] calculates the leading bias in the maximum-likelihood estimator of ϑ and considers the effectiveness of subtracting a plug-in estimator of it from the maximum-likelihood estimator. We will compare this approach to estimation based on the modified likelihood in numerical experiments below.

4.1. *Modified profile likelihood.* The likelihood function, conditional on the regressors, is

$$\ell(\vartheta, \beta) = \sum_{i=1}^{n} \sum_{i<j} y_{ij} \, \log\left(F_{ij}(\vartheta, \beta_i, \beta_j)\right) + (1 - y_{ij}) \, \log\left(1 - F_{ij}(\vartheta, \beta_i, \beta_j)\right),$$

where we let $F_{ij}(\vartheta, \beta_i, \beta_j) = F(\beta_i + \beta_j + x_{ij}'\vartheta)$.

For a given value of ϑ, the score for the incidental parameters has elements

$$\frac{\partial \ell(\vartheta, \beta)}{\partial \beta_i} = \sum_{i<j} y_{ij} - F_{ij}(\vartheta, \beta_i, \beta_j) + \sum_{i>j} y_{ji} - F_{ji}(\vartheta, \beta_j, \beta_i)$$

while the $n \times n$ Hessian matrix has (i,j)th-entry equal to

$$\frac{\partial^2 \ell(\vartheta, \beta)}{\partial \beta_i \partial \beta_j} = \begin{cases} -\sum_{i<k} f_{ik}(\vartheta, \beta_i, \beta_k) - \sum_{i>k} f_{ki}(\vartheta, \beta_k, \beta_i) & \text{if } i = j \\ -f_{ij}(\vartheta, \beta_i, \beta_j) & \text{if } i < j \\ -f_{ji}(\vartheta, \beta_j, \beta_i) & \text{if } i > j \end{cases},$$

for $f_{ij}(\vartheta, \beta_i, \beta_j) = F_{ij}(\vartheta, \beta_i, \beta_j)\,(1 - F_{ij}(\vartheta, \beta_i, \beta_j))$. The maximum-likelihood estimator of the β_i for a given value of ϑ is not available in closed form and needs to be computed numerically. Because the likelihood is globally concave, Newton's algorithm is well-suited for the task, and will typically find the solution in two or three iterations.

Introduce the shorthands

$$\hat{F}_{ij}(\vartheta) = F_{ij}(\vartheta, \hat{\beta}_i(\vartheta), \hat{\beta}_j(\vartheta)), \quad \hat{f}_{ij}(\vartheta) = f_{ij}(\vartheta, \hat{\beta}_i(\vartheta), \hat{\beta}_j(\vartheta)).$$

The profile likelihood is

$$\hat{\ell}(\vartheta) = \sum_{i=1}^{n} \sum_{i<j} y_{ij} \, \log(\hat{F}_{ij}(\vartheta)) + (1 - y_{ij}) \, \log(1 - \hat{F}_{ij}(\vartheta)),$$

and a modified likelihood is readily constructed by appropriately combining the matrices $\hat{\Sigma}(\vartheta)$ and $\hat{\Omega}(\vartheta)$.

4.2. *Simulation experiments.* We next present the results from a Monte Carlo experiment. The designs are borrowed from Graham [2017]. All designs are of the following form. Let $u_i \in \{-1, 1\}$ so that $\mathbb{P}(u_i = 1) = \frac{1}{2}$. We generate the dyad covariate as

$$x_{ij} = u_i \, u_j,$$

and the fixed effects as

$$\beta_i = \mu + \gamma_1 \frac{1 + u_i}{2} + \gamma_2 \frac{1 - u_i}{2} + v_i,$$

where $v_i \sim \text{Beta}(\lambda_1, \lambda_2)$. We set $\mu = -\lambda_1(\lambda_1 + \lambda_2)^{-1}$, so that $\mu + v_i$ has mean zero, and will consider several choices for the parameters (γ_1, γ_2) and (λ_1, λ_2). The parameter choices are summarized in Table 2. In the first four designs (A1–A4), the β_i are drawn independently of x_{ij} from symmetric Beta distributions. In the next four designs (B1–B4) the β_i are generated from skewed distributions that depend on u_i (and thus correlate with the regressor x_{ij}). For both the designs labelled A and B, the average number of observed links per agent goes down as we move from the first design (A1 and B1) to the fourth design (A4 and B4). The average number of links decreases from about 50% to 12%. This is clear from the second block of Table 2, which contains the average, minimum, and maximum number of links per agent (in percentages).

Table 2
Simulation designs for the β-model

Design	γ_1	γ_2	λ_1	λ_2	degree (%)		
					mean	min	max
A1	0	0	1	1	50	32	67
A2	-0.25	-0.25	1	1	40	24	57
A3	-0.75	-0.75	1	1	23	10	38
A4	-1.25	-1.25	1	1	12	3	22
B1	0	0.50	0.25	0.75	60	40	78
B2	-0.50	0	0.25	0.75	40	21	62
B3	-1.00	-0.50	0.25	0.75	24	8	44
B4	-1.50	-1.00	0.25	0.75	12	2	28

We simulate $10,000$ data sets for each design for $n \in \{25, 50, 75, 100\}$ and $\vartheta = 1$. Because the results across designs are qualitatively very similar, we present the full set of results only for Design A1 (Table 3). Tables 4 and 5 provide the results for $n \in \{50, 100\}$ for all designs. Each table contains the mean and median bias of ϑ, $\dot{\vartheta}$, and $\ddot{\vartheta}$, along with their standard deviation and their interquartile range (both across the Monte Carlo replications). The tables also provide the empirical size of the likelihood ratio test for the null that $\vartheta = 1$ for theoretical size $\alpha \in \{.05, .10\}$. Inference results based on the Wald statistic, using a plug-in estimator of $I(\vartheta)$, are very similar and not reported for brevity.

Because the results for $n = 100$ can be compared (up to Monte Carlo error) to the numerical results collected in Graham [2017, Table 2], Table 5 contains two additional columns in which we reproduce the results for his analytically bias-corrected maximum-likelihood estimator ($\tilde{\vartheta}$) and his 'tetrad logit' estimator ($\check{\vartheta}$). The latter is based on moment conditions that are free of β_i using a sufficiency argument. Bias correcting $\hat{\vartheta}$ does not salvage the likelihood ratio statistic, and the conditional likelihood function of the 'tetrad logit' estimator is a quasi likelihood and, therefore, does not satisfy the information equality. Hence, the results on size for these two estimators are based on the Wald statistic.

Table 3 clearly shows that both the bias and standard deviation of $\hat{\vartheta}$ are $O(n^{-1})$. Consequently, the likelihood ratio test is size distorted even in large samples. Point estimation through the modified likelihoods gives estimators with small bias relative to their standard error. Even for $n = 25$, the bias is only about 20% of the bias in maximum likelihood estimator. In larger

Table 3
β-model. Design A1 for all n

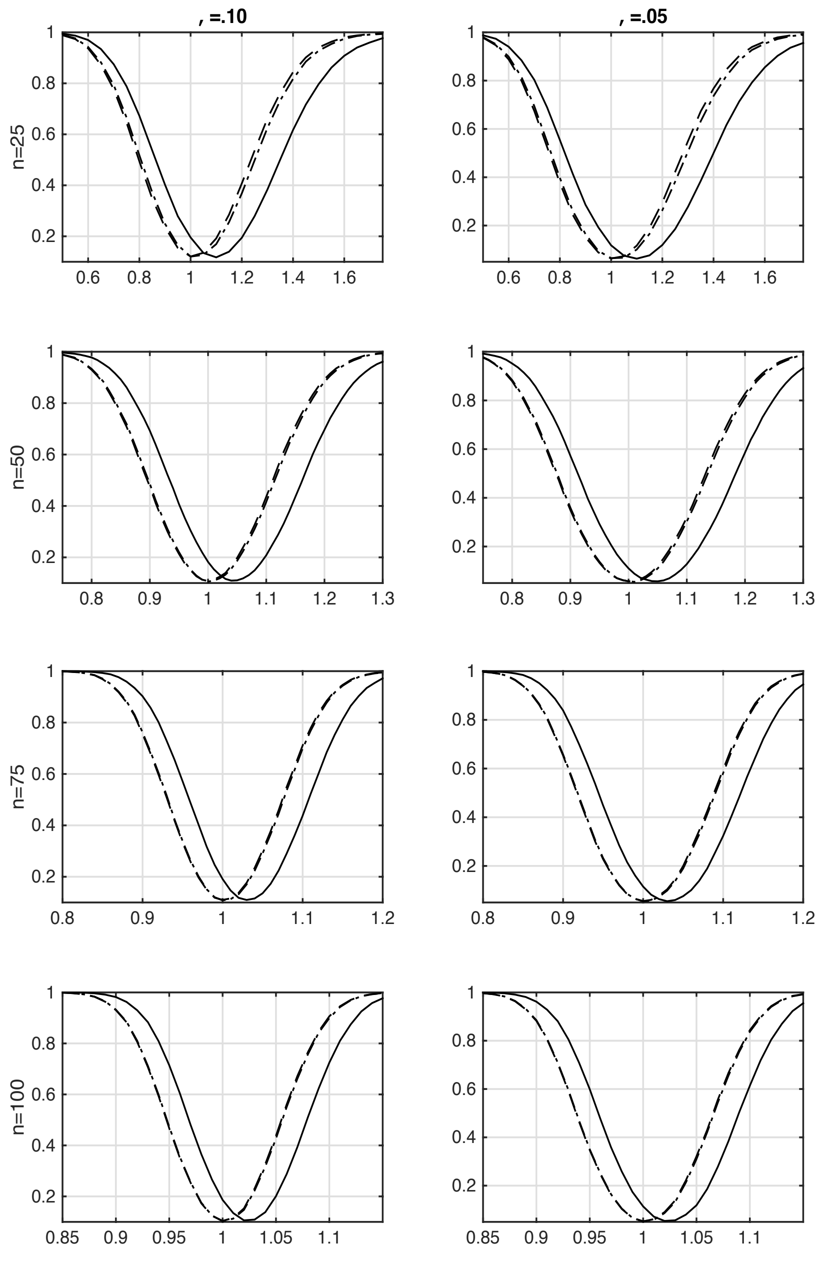
n	$\hat{\vartheta}$	$\dot{\vartheta}$	$\ddot{\vartheta}$	$\hat{\vartheta}$	$\dot{\vartheta}$	$\ddot{\vartheta}$
	mean bias			standard deviation		
25	0.1098	0.0204	0.0304	0.1897	0.1560	0.1572
50	0.0492	0.0045	0.0071	0.0717	0.0679	0.0681
75	0.0320	0.0020	0.0032	0.0467	0.0450	0.0451
100	0.0237	0.0011	0.0017	0.0341	0.0332	0.0332
	median bias			interquartile range		
25	0.1029	0.0154	0.0253	0.2069	0.1873	0.1889
50	0.0487	0.0042	0.0067	0.0961	0.0913	0.0914
75	0.0316	0.0017	0.0028	0.0630	0.0607	0.0608
100	0.0236	0.0010	0.0017	0.0464	0.0450	0.0451
	empirical size ($\alpha = .10$)			empirical size ($\alpha = .05$)		
25	0.1937	0.1134	0.1147	0.1142	0.0627	0.0637
50	0.1896	0.1128	0.1125	0.1178	0.0558	0.0555
75	0.1866	0.1092	0.1081	0.1142	0.0575	0.0569
100	0.1890	0.1042	0.1025	0.1103	0.0520	0.0513

samples, the estimators are essentially unbiased. Interestingly, both $\dot{\vartheta}$ and $\ddot{\vartheta}$ are also less volatile than is $\hat{\vartheta}$. Thus, here, bias correction does not come at the cost of an increase in dispersion. Together with the substantial decrease in mean squared error, inference, too, improves dramatically. The likelihood ratio statistics for $\dot{\ell}(\vartheta)$ and $\ddot{\ell}(\vartheta)$ have near-theoretical size for all n.

To give a more complete picture on inference via modifying the profile likelihood Figure 1 presents power curves for the likelihood ratio statistic that go along with Table 3. The curves for $\hat{\ell}(\vartheta)$ (solid lines) are symmetric but not correctly centered, reflecting the fact that they are size distorted. This is so for all sample sizes and significance levels considered. Modifying the likelihood shifts the power curve so that the likelihood ratio test is (approximately) size correct. This is done without significantly altering the shape of the power curves. For the smallest sample size considered ($n = 25$; upper two plots) there is a small difference in power between the likelihood ratio test for $\dot{\ell}(\vartheta)$ (dashed lines) and $\ddot{\ell}(\vartheta)$ (dashed-dotted lines); the former has slightly higher power than the latter for alternatives $\vartheta > 1$, and slightly less power for $\vartheta < 1$. This difference vanished rapidly as n increases, however, which is in line with the similar performance of both corrections observed in Table 3.

Tables 4 and 5 show that all conclusions from Design A1 carry over to the

FIG 1. *Power curves. Design A1 for all n*



Power curves for likelihood ratio statistic based on $\hat{\ell}(\vartheta)$ (solid lines), $\dot{\ell}(\vartheta)$ (dashed lines), $\ddot{\ell}(\vartheta)$ (dashed-dotted lines).

TABLE 4
β-model. All designs for $n = 50$

Design	$\hat{\vartheta}$	$\dot{\vartheta}$	$\ddot{\vartheta}$	$\hat{\vartheta}$	$\dot{\vartheta}$	$\ddot{\vartheta}$
		mean bias			standard deviation	
A1	0.0492	0.0045	0.0071	0.0717	0.0679	0.0681
A2	0.0499	0.0054	0.0079	0.0742	0.0704	0.0705
A3	0.0467	0.0033	0.0047	0.0933	0.0890	0.0891
A4	0.0497	0.0049	0.0024	0.1391	0.1335	0.1335
B1	0.0526	0.0073	0.0096	0.0768	0.0728	0.0729
B2	0.0490	0.0035	0.0059	0.0747	0.0707	0.0708
B3	0.0493	0.0046	0.0060	0.0936	0.0891	0.0891
B4	0.0500	0.0043	0.0005	0.1380	0.1320	0.1316
		median bias			interquartile range	
A1	0.0487	0.0042	0.0067	0.0961	0.0913	0.0914
A2	0.0482	0.0040	0.0064	0.0995	0.0943	0.0945
A3	0.0441	0.0008	0.0022	0.1247	0.1191	0.1191
A4	0.0412	-0.0032	-0.0059	0.1827	0.1748	0.1748
B1	0.0513	0.0061	0.0084	0.1034	0.0981	0.0982
B2	0.0479	0.0024	0.0049	0.0999	0.0948	0.0949
B3	0.0470	0.0024	0.0039	0.1252	0.1195	0.1196
B4	0.0438	-0.0018	-0.0052	0.1827	0.1740	0.1743
		empirical size ($\alpha = .10$)			empirical size ($\alpha = .05$)	
A1	0.1896	0.1128	0.1125	0.1178	0.0558	0.0555
A2	0.1857	0.1135	0.1118	0.1139	0.0602	0.0603
A3	0.1565	0.1098	0.1082	0.0878	0.0581	0.0563
A4	0.1287	0.1095	0.1083	0.0664	0.0594	0.0592
B1	0.1902	0.1141	0.1112	0.1146	0.0582	0.0579
B2	0.1801	0.1081	0.1049	0.1040	0.0574	0.0564
B3	0.1498	0.1052	0.1030	0.0830	0.0554	0.0538
B4	0.1236	0.1064	0.1067	0.0634	0.0543	0.0551

TABLE 5. β-*model. All designs for* $n = 100$

Design	$\hat{\vartheta}$	$\dot{\vartheta}$	$\ddot{\vartheta}$	$\tilde{\vartheta}$	$\check{\vartheta}$	$\hat{\vartheta}$	$\dot{\vartheta}$	$\ddot{\vartheta}$	$\tilde{\vartheta}$	$\check{\vartheta}$
	mean bias					standard deviation				
A1	0.0237	0.0011	0.0017	—	—	0.0341	0.0332	0.0332	0.0340	0.0340
A2	0.0240	0.0015	0.0022	—	—	0.0352	0.0343	0.0343	0.0350	0.0350
A3	0.0215	0.0003	0.0006	—	—	0.0444	0.0434	0.0434	0.0460	0.0440
A4	0.0206	0.0001	-0.0004	—	—	0.0645	0.0633	0.0633	0.0790	0.0650
B1	0.0241	0.0013	0.0019	—	—	0.0356	0.0346	0.0346	0.0360	0.0370
B2	0.0246	0.0015	0.0021	—	—	0.0359	0.0349	0.0349	0.0340	0.0340
B3	0.0222	0.0004	0.0007	—	—	0.0450	0.0439	0.0439	0.0470	0.0460
B4	0.0231	0.0020	0.0015	—	—	0.0651	0.0639	0.0639	0.0810	0.0670
	median bias					interquartile range				
A1	0.0236	0.0010	0.0017	0.0010	-0.0010	0.0464	0.0450	0.0451	—	—
A2	0.0234	0.0010	0.0017	0.0020	0.0000	0.0471	0.0458	0.0458	—	—
A3	0.0199	-0.0013	-0.0010	0.0190	0.0010	0.0598	0.0584	0.0584	—	—
A4	0.0181	-0.0022	-0.0028	0.1060	0.0050	0.0870	0.0854	0.0855	—	—
B1	0.0240	0.0012	0.0019	0.0040	0.0010	0.0482	0.0469	0.0469	—	—
B2	0.0247	0.0017	0.0022	0.0030	0.0010	0.0484	0.0472	0.0472	—	—
B3	0.0213	-0.0007	-0.0003	0.0170	0.0000	0.0604	0.0591	0.0591	—	—
B4	0.0212	0.0002	-0.0003	0.0860	0.0020	0.0867	0.0852	0.0851	—	—
	empirical size ($\alpha = .10$)					empirical size ($\alpha = .05$)				
A1	0.1890	0.1042	0.1025	0.1020	0.0760	0.1103	0.0520	0.0513	0.0550	0.0360
A2	0.1846	0.1044	0.1026	0.0970	0.0810	0.1081	0.0533	0.0526	0.0570	0.0420
A3	0.1447	0.1051	0.1037	0.1260	0.0800	0.0805	0.0522	0.0515	0.0680	0.0380
A4	0.1149	0.1003	0.1002	0.4670	0.0810	0.0593	0.0505	0.0500	0.3520	0.0360
B1	0.1748	0.1002	0.0978	0.1050	0.0810	0.0999	0.0494	0.0488	0.0610	0.0340
B2	0.1846	0.1064	0.1050	0.0870	0.0610	0.1089	0.0533	0.0521	0.0350	0.0250
B3	0.1437	0.1050	0.1036	0.1400	0.0850	0.0812	0.0522	0.0516	0.0760	0.0380
B4	0.1174	0.1007	0.1003	0.3910	0.0900	0.0621	0.0509	0.0507	0.2960	0.0430

other designs. Moreover, the introduction of correlation between regressors and heterogenous coefficients or skewing the distribution from which the latter are drawn does not prevent the modified likelihood to improve on maximum likelihood both in terms of point estimation and inference. A comparison of the two tables clearly shows that both the bias and standard deviation of $\hat{\vartheta}$ shrink by a factor of one half as n doubles, again illustrating that both are of order n^{-1}. The subsequent reduction in bias by considering $\dot{\vartheta}$ and $\ddot{\vartheta}$ and improvement in size are manifest for all designs.

Table 5 further shows that the modified-likelihood approach outperforms bias correction of the maximum-likelihood estimator in Designs A3 and B3 and, in particular, in Designs A4 and B4. There, bias correction of maximum likelihood introduces rather substantial additionial bias relative to $\hat{\vartheta}$. The additional bias also leads to a large deterioration of the empirical size of the Wald statistic associated with $\check{\vartheta}$, with actual sizes ranging up to seven times the nominal size. The performance of the modified likelihood is comparable to Graham's 'tetrad logit' estimator $\check{\vartheta}$ in terms of bias, and it tends to be somewhat more accurate in terms of the empirical size of the associated hypothesis tests. Moreover, inference based on the 'tetrad logit' estimator is conservative in all designs even though, with $n = 100$ and therefore $4,950$ dyadic observations, the sample size is large.

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