# Joint Approximate Asymmetric Diagonalization by Non-orthogonal Matrices

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**Abstract**

This paper introduces an algorithm, JASYD, to perform what we call approximate joint asymmetric diagonalization of a collection of non-orthogonal matrices. This problem is a natural generalisation of joint (or simultaneous) diagonalization and appears not to have been studied previously. Our method is based on the polar decomposition and uses a Jacobi-like iterative scheme to sequentially optimise a series of suitable criteria. We find JASYD to be both an effective and efficient way to compute a solution to this problem.

# 1 Introduction

## 1.1 Joint Asymmetric Diagonalization

Consider a collection of real $N \times N$ matrices $\mathscr{M} \coloneqq \{\boldsymbol{M}_{k\kappa} \text{ for } k, \kappa = 1, \ldots, K\}$. We study the problem of finding a set of non-singular matrices $\boldsymbol{V}_1, \ldots, \boldsymbol{V}_K$ which satisfy

$$\boldsymbol{V}_k^{-1} \boldsymbol{M}_{k\kappa} \boldsymbol{V}_\kappa = \boldsymbol{D}_{k\kappa} \text{ for } k, \kappa = 1, \ldots, K, \tag{1}$$

where $\boldsymbol{D}_{k\kappa}$ are unknown diagonal matrices. This problem is a generalisation of matrix diagonalization, which we term *asymmetric diagonalization* due to the fact that, as well as symmetric restrictions of the form $\boldsymbol{V}_k^{-1} \boldsymbol{M}_{kk} \boldsymbol{V}_k = \boldsymbol{D}_{kk}$, we also have asymmetric restrictions of the form $\boldsymbol{V}_k^{-1} \boldsymbol{M}_{k\kappa} \boldsymbol{V}_\kappa = \boldsymbol{D}_{k\kappa}$ for $k \neq \kappa$. We may extend this problem to the situation

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where $M$ collections of matrices $\mathscr{M}^{(1)}, \ldots, \mathscr{M}^{(M)}$ are available. In this case we refer to the problem of finding a set of common matrices $\boldsymbol{V}_1, \ldots, \boldsymbol{V}_K$ that asymmetrically diagonalize all $M$ collections of matrices as *joint asymmetric diagonalization*. This, naturally, is a generalisation of joint (or simultaneous) diagonalization which, to the best of our knowledge, has not been studied previously. In this paper we introduce a novel algorithm called JASYD to compute a solution to this problem. Following the success of Jacobi-like iterative schemes in performing joint diagonalization by congruence (Cardoso and Souloumiac, 1996), and joint diagonalization (Iferroudjene et al., 2009; Luciani and Albera, 2010), we adopt a similar approach to joint asymmetric diagonalization. In particular, we follow Luciani and Albera (2010) and use the polar decomposition to construct our algorithm. We find our method to be a quick and effective means to compute an exact solution to (1), where such a solution exists, and of otherwise finding an approximate solution.

### 1.2 Notation

The operation $\text{diag}(\cdot)$ applied to an $N \times N$ matrix $\boldsymbol{A}$ produces an $N \times N$ matrix $\text{diag}(\boldsymbol{A})$ which contains the diagonal elements of $\boldsymbol{A}$ along its diagonal. Complementary to this is the operation $\text{off}(\boldsymbol{A}) := \boldsymbol{A} - \text{diag}(\boldsymbol{A})$. Let $\mathscr{A} := \{\boldsymbol{A}_{kx} : k, x = 1, \ldots, K\}$ be a collection of $N \times N$ matrices. The $NK \times NK$ block matrix $\boldsymbol{\mathcal{A}}$ is defined as

$$\boldsymbol{\mathcal{A}} := \begin{pmatrix} \boldsymbol{A}_{11} & \cdots & \boldsymbol{A}_{1K} \\ \vdots & \ddots & \vdots \\ \boldsymbol{A}_{K1} & \cdots & \boldsymbol{A}_{KK} \end{pmatrix}.$$

The operation $\text{blkdiag}(\cdot)$ applied to a block matrix sets all of the off-diagonal blocks to zero, and the operation $\text{bon}(\cdot)$ applied to a block matrix with blocks of equal size produces a matrix containing only the diagonal elements of the blocks, e.g.

$$\text{blkdiag}(\boldsymbol{\mathcal{A}}) := \begin{pmatrix} \boldsymbol{A}_{11} & \cdots & \boldsymbol{0}_{N\times N} \\ \vdots & \ddots & \vdots \\ \boldsymbol{0}_{N\times N} & \cdots & \boldsymbol{A}_{KK} \end{pmatrix}, \quad \text{bon}(\boldsymbol{\mathcal{A}}) := \begin{pmatrix} \text{diag}(\boldsymbol{A}_{11}) & \cdots & \text{diag}(\boldsymbol{A}_{1K}) \\ \vdots & \ddots & \vdots \\ \text{diag}(\boldsymbol{A}_{K1}) & \cdots & \text{diag}(\boldsymbol{A}_{KK}) \end{pmatrix}.$$

Analogously, the operation $\text{boff}(\cdot)$ applied to a block matrix with blocks of equal size produces a matrix containing only the off-diagonal elements of the blocks, e.g. $\text{boff}(\boldsymbol{\mathcal{A}}) := \boldsymbol{\mathcal{A}} - \text{bon}(\boldsymbol{\mathcal{A}})$. Givens and Hyperbolic rotation matrices are denoted by $\boldsymbol{G}(\theta_{t\tau})$ and $\boldsymbol{H}(\phi_{t\tau})$, respectively, which are equal to an identity matrix except for the following elements:

$$\begin{aligned} \boldsymbol{G}(\theta_{tj})_{tt} &= \cos(\theta_{t\tau}), & \boldsymbol{H}(\phi_{tj})_{tt} &= \cosh(\phi_{t\tau}), \\ \boldsymbol{G}(\theta_{tj})_{t\tau} &= -\sin(\theta_{t\tau}), & \boldsymbol{H}(\phi_{tj})_{t\tau} &= \sinh(\phi_{t\tau}), \\ \boldsymbol{G}(\theta_{tj})_{\tau t} &= \sin(\theta_{t\tau}), & \boldsymbol{H}(\phi_{tj})_{\tau t} &= \sinh(\phi_{t\tau}), \\ \boldsymbol{G}(\theta_{tj})_{\tau\tau} &= \cos(\theta_{t\tau}), & \boldsymbol{H}(\phi_{tj})_{\tau\tau} &= \cosh(\phi_{t\tau}). \end{aligned}$$

# 2 JASYD

## 2.1 Iterative Scheme

For ease of exposition it is useful to first define the $NK \times NK$ block matrices

$$\boldsymbol{\mathcal{M}}^{(m)} := \begin{pmatrix} \boldsymbol{M}_{11}^{(m)} & \cdots & \boldsymbol{M}_{1K}^{(m)} \\ \vdots & \ddots & \vdots \\ \boldsymbol{M}_{K1}^{(m)} & \cdots & \boldsymbol{M}_{KK}^{(m)} \end{pmatrix}, \quad \boldsymbol{\mathcal{D}}^{(m)} := \begin{pmatrix} \boldsymbol{D}_{11}^{(m)} & \cdots & \boldsymbol{D}_{1K}^{(m)} \\ \vdots & \ddots & \vdots \\ \boldsymbol{D}_{K1}^{(m)} & \cdots & \boldsymbol{D}_{KK}^{(m)} \end{pmatrix},$$

and

$$\boldsymbol{\mathcal{V}} := \begin{pmatrix} \boldsymbol{V}_1 & \cdots & \boldsymbol{0}_{N\times N} \\ \vdots & \ddots & \vdots \\ \boldsymbol{0}_{N\times N} & \cdots & \boldsymbol{V}_K, \end{pmatrix},$$

whereby we may recast the problem of joint asymmetric diagonalization as computing a block diagonal matrix $\boldsymbol{\mathcal{V}}$ that solves

$$\boldsymbol{\mathcal{V}}^{-1}\boldsymbol{\mathcal{M}}^{(m)}\boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{D}}^{(m)} \text{ for } m = 1, \dots, M.^{1} \tag{2}$$

By the polar decomposition, any non-singular real matrix can be factorised into the product of a symmetric matrix and an orthogonal matrix. Therefore we can decompose $\boldsymbol{\mathcal{V}} =: \boldsymbol{S}\boldsymbol{\mathcal{Q}}$, where the block diagonal matrices $\boldsymbol{S}$ and $\boldsymbol{\mathcal{Q}}$ are symmetric and orthogonal, respectively. Furthermore, the symmetric matrix $\boldsymbol{S}$ can be decomposed into the product of Hyperbolic rotation matrices and a diagonal matrix, while the orthogonal matrix $\boldsymbol{\mathcal{Q}}$ can be decomposed into the product of Givens rotation matrices and an orthogonal diagonal matrix. Thus we may decompose

$$\begin{aligned} \boldsymbol{\mathcal{V}} = \boldsymbol{S}\boldsymbol{\mathcal{Q}} &= \prod_{k=1}^{K}\prod_{i=1}^{N-1}\prod_{j=i+1}^{N} \boldsymbol{H}(\phi_{k,ij})\boldsymbol{G}(\theta_{k,ij})\boldsymbol{\Omega} \\ &= \prod_{r=1}^{R} \boldsymbol{H}(\phi_r)\boldsymbol{G}(\theta_r)\boldsymbol{\Omega}, \end{aligned}$$

where $\boldsymbol{\Omega}$ is an $NK \times NK$ diagonal matrix and, the index $r = 1, \ldots, R$ with $R \coloneqq KN(N-1)/2$ corresponds to a pair of parameters $\{\theta_{k,ij}, \phi_{k,ij}\}$ which relate to element $(i,j)$ in the $(k,k)$-th block of $\boldsymbol{\mathcal{V}}$. Hence we may re-write (2) as

$$\left(\prod_{r=1}^{R} \boldsymbol{G}^{\top}(\theta_r)\boldsymbol{H}^{-1}(\phi_r)\right)\boldsymbol{\mathcal{M}}^{(m)}\left(\prod_{r=1}^{R} \boldsymbol{H}(\phi_r)\boldsymbol{G}(\theta_r)\right) = \boldsymbol{\mathcal{D}}^{(m)},$$

where we retain the notation $\boldsymbol{\mathcal{D}}^{(m)}$ for a generic matrix of the corresponding structure.

Since Given and Hyperbolic rotation matrices are defined by only a single parameter, we sequentially compute the $R$ parameters which produce a matrix $\boldsymbol{\mathcal{D}}^{(m)}$ for all $m$. This is achieved by computing, at each step, optimal parameter values that minimise two suitable criteria, denoted by $f_H(\phi_r)$ and $f_G(\theta_r)$, respectively. This gives rise to the following algorithm.

**Algorithm:** JASYD(a)

**Result:** Approximate joint asymmetric diagonalization of a collection of matrices.
Set $r = 0$, $\boldsymbol{\mathcal{M}}^{(m,0)} = \boldsymbol{\mathcal{M}}^{(m)}$ for $m = 1, \ldots, M$ and $\boldsymbol{\mathcal{V}}^{(0)} = \boldsymbol{I}_{NK}$.
**for** $r = 1, \ldots, R$ **do**
  $\phi_r^* \coloneqq \arg\min f_H(\boldsymbol{\mathcal{M}}^{(1,r-1)}, \ldots, \boldsymbol{\mathcal{M}}^{(M,r-1)}; \phi)$
  $\tilde{\boldsymbol{\mathcal{V}}}^{(r)} \coloneqq \boldsymbol{\mathcal{V}}^{(r-1)}\boldsymbol{H}(\phi_r^*)$
  **for** $m = 1, \ldots, M$ **do**
    $\tilde{\boldsymbol{\mathcal{M}}}^{(m,r)} \coloneqq (\tilde{\boldsymbol{\mathcal{V}}}^{(r)})^{-1}\boldsymbol{\mathcal{M}}^{(m,r-1)}\tilde{\boldsymbol{\mathcal{V}}}^{(r)}$
  **end**
  $\theta_r^* \coloneqq \arg\min f_G(\tilde{\boldsymbol{\mathcal{M}}}^{(1,r)}, \ldots, \tilde{\boldsymbol{\mathcal{M}}}^{(M,r)}; \theta)$
  $\boldsymbol{\mathcal{V}}^{(r)} \coloneqq \tilde{\boldsymbol{\mathcal{V}}}^{(r)}\boldsymbol{G}(\theta_r^*)$
  **for** $m = 1, \ldots, M$ **do**
    $\boldsymbol{\mathcal{M}}^{(m,r)} \coloneqq (\boldsymbol{\mathcal{V}}^{(r)})^{-1}\tilde{\boldsymbol{\mathcal{M}}}^{(m,r-1)}\boldsymbol{\mathcal{V}}^{(r)}$
  **end**
**end**

In order to achieve convergence, the above procedure should be repeated several times and terminate when a suitable degree of diagonalization is achieved. An obvious criterion by

[1]Motivated by the joint asymmetric diagonalization problem we focus on the case in which $\boldsymbol{\mathcal{V}}$ is block diagonal. It is, however, possible to extend our method to the case where the structure of $\boldsymbol{\mathcal{V}}$ is left unrestricted.

which to measure this is

$$\sum_{m=1}^{M} ||\text{boff}(\boldsymbol{\mathcal{M}}^{(m,R)})||_F^2.$$

### 2.2 Computing the Hyperbolic rotation parameters

Let

$$\boldsymbol{\mathcal{B}}^{(m,r)} := \boldsymbol{H}^{-1}(\phi)\boldsymbol{\mathcal{M}}^{(m,r)}\boldsymbol{H}(\phi).$$

In order to compute the optimal angle of the Hyperbolic rotations, we may adopt the approach outlined in Luciani and Albera (2010). This involves targetting certain elements of $\boldsymbol{\mathcal{B}}^{(m,r)}$ that are transformed twice by the Hyperbolic rotations. Recall that each $r \mapsto (k, i, j)$. We minimise

$$f_H(\boldsymbol{\mathcal{M}}^{(1,r)}, \ldots, \boldsymbol{\mathcal{M}}^{(M,r)}; \phi) := \sum_{m=1}^{M} (\boldsymbol{B}_{kk,ii}^{(m)})^2 + (\boldsymbol{B}_{kk,jj}^{(m)})^2,$$

where $\boldsymbol{B}_{k\kappa,ij}^{(m)}$ corresponds the $(i, j)$-th element in the $(k, \kappa)$-th block of $\boldsymbol{\mathcal{B}}^{(r,m)}$. A closed form solution for the optimal angle $\phi^*$ is provided by Luciani and Albera (2010).

### 2.3 Computing the Givens rotation parameters

An obvious criterion to use for computing the $r$-th Givens angle is to minimise

$$\sum_{m=1}^{M} ||\text{boff}(\boldsymbol{G}^\top(\theta)\tilde{\boldsymbol{\mathcal{M}}}^{(m,r)}\boldsymbol{G}(\theta))||_F^2 =: ||\text{boff}(\tilde{\boldsymbol{\mathcal{C}}}^{(m,r)})||_F^2. \tag{3}$$

Similar sum-of-squares criteria have been used successfully in other contexts. However, rather than seek to optimise (3), we opt instead to optimise an analogue of this objective function which places relatively less weight on the diagonal elements. Our reasons for doing so will become clear shortly. Nonetheless, we begin by deriving an expression for the minimiser of (3) in terms of the elements of the matrix $\tilde{\boldsymbol{\mathcal{M}}}^{(m,r)}$. To ease notation, in the following we fix $m$ and suppress the dependence of both $\tilde{\boldsymbol{\mathcal{M}}}^{(m,r)}$ and $\tilde{\boldsymbol{\mathcal{C}}}^{(m,r)}$ on this quantity. Since the Frobenius norm is invariant to orthogonal transformations,

$$||\tilde{\boldsymbol{\mathcal{C}}}^{(r)}||_F^2 = ||\tilde{\boldsymbol{\mathcal{M}}}^{(r)}||_F^2,$$

and so

$$||\text{bon}(\tilde{\boldsymbol{\mathcal{C}}}^{(r)})||_F^2 + ||\text{boff}(\tilde{\boldsymbol{\mathcal{C}}}^{(r)})||_F^2 = ||\text{bon}(\tilde{\boldsymbol{\mathcal{M}}}^{(r)})||_F^2 + ||\text{boff}(\tilde{\boldsymbol{\mathcal{M}}}^{(r)})||_F^2.$$

Recall that each $r \mapsto (k, i, j)$. Then

$$\begin{aligned} ||\text{bon}(\tilde{\boldsymbol{\mathcal{C}}}^{(r)})||_F^2 &= \sum_{k'=1}^{K}\sum_{\kappa'=1}^{K}\sum_{i'=1}^{n} \tilde{\boldsymbol{C}}^2_{k'\kappa',i'i'} \\ &= \sum_{k'=1}^{K}(\tilde{\boldsymbol{C}}^2_{k'k,ii} + \tilde{\boldsymbol{C}}^2_{k'k,jj} + \tilde{\boldsymbol{C}}^2_{kk',ii} + \tilde{\boldsymbol{C}}^2_{kk',jj}) - \tilde{\boldsymbol{C}}^2_{kk,ii} - \tilde{\boldsymbol{C}}^2_{kk,jj} + \text{rem}(\tilde{\boldsymbol{\mathcal{C}}}^{(r)}), \end{aligned}$$

where $\tilde{\boldsymbol{C}}_{k\kappa,ij}$ corresponds the $(i, j)$-th element in the $(k, \kappa)$-th block of $\tilde{\boldsymbol{\mathcal{C}}}^{(r)}$. Thus,

$$\begin{aligned} ||\text{boff}(\tilde{\boldsymbol{\mathcal{C}}}^{(r)})||_F^2 &= ||\text{bon}(\tilde{\boldsymbol{\mathcal{M}}})^{(r)}||_F^2 + ||\text{boff}(\tilde{\boldsymbol{\mathcal{M}}}^{(r)})||_F^2 - \text{rem}(\tilde{\boldsymbol{\mathcal{C}}}^{(r)}) \\ &\quad - \left(\sum_{k'=1}^{K}\tilde{\boldsymbol{C}}^2_{k'k,ii} + \sum_{k'=1}^{K}\tilde{\boldsymbol{C}}^2_{k'k,jj} + \sum_{k'=1}^{K}\tilde{\boldsymbol{C}}^2_{kk',ii} + \sum_{k'=1}^{K}\tilde{\boldsymbol{C}}^2_{kk',jj} - \tilde{\boldsymbol{C}}^2_{kk,ii} - \tilde{\boldsymbol{C}}^2_{kk,jj}\right). \end{aligned} \tag{4}$$

Notice that the elements of $\tilde{\boldsymbol{\mathcal{M}}}^{(r)}$ and $\tilde{\boldsymbol{\mathcal{C}}}^{(r)}$ are the same, apart from those in the $i$-th and $j$-th row and column, and therefore the elements in $\text{rem}(\tilde{\boldsymbol{\mathcal{C}}}^{(r)})$ do not depend on the parameter $\theta$. Hence, the first three terms on the right-hand side of (4) do not depend on $\theta$ and so minimising $||\text{boff}(\tilde{\boldsymbol{\mathcal{C}}}^{(r)})||_F^2$ with respect to this parameter is equivalent to maximising the negative of the last term on the right. After some rearranging we find that

$$\begin{aligned} &\sum_{k'=1}^{K}\tilde{\boldsymbol{C}}^2_{k'k,ii} + \sum_{k'=1}^{K}\tilde{\boldsymbol{C}}^2_{k'k,jj} + \sum_{k'=1}^{K}\tilde{\boldsymbol{C}}^2_{kk',ii} + \sum_{k'=1}^{K}\tilde{\boldsymbol{C}}^2_{kk',jj} - \tilde{\boldsymbol{C}}^2_{kk,ii} - \tilde{\boldsymbol{C}}^2_{kk,jj} \\ &= \tilde{\boldsymbol{C}}^2_{kk,ii} + \tilde{\boldsymbol{C}}^2_{kk,jj} + \sum_{k'\neq k}^{K}(\tilde{\boldsymbol{C}}^2_{k'k,ii} + \tilde{\boldsymbol{C}}^2_{kk',ii}) + \sum_{k'\neq k}^{K}(\tilde{\boldsymbol{C}}^2_{k'k,jj} + \tilde{\boldsymbol{C}}^2_{kk',jj}). \end{aligned}$$

Using $a^2 + b^2 = (a + b)^2/2 + (a - b)^2/2$, this becomes

$$\begin{aligned} &(\tilde{\boldsymbol{C}}_{kk,ii} + \tilde{\boldsymbol{C}}_{kk,jj})^2 + (\tilde{\boldsymbol{C}}_{kk,ii} - \tilde{\boldsymbol{C}}_{kk,jj})^2 + \sum_{k'\neq k}^{K}(\tilde{\boldsymbol{C}}_{k'k,ii} + \tilde{\boldsymbol{C}}_{kk',ii})^2 + \sum_{k'\neq k}^{K}(\tilde{\boldsymbol{C}}_{k'k,ii} - \tilde{\boldsymbol{C}}_{kk',ii})^2 \\ &\quad + \sum_{k'\neq k}^{K}(\tilde{\boldsymbol{C}}_{k'k,jj} + \tilde{\boldsymbol{C}}_{kk',jj})^2 + \sum_{k'\neq k}^{K}(\tilde{\boldsymbol{C}}_{k'k,jj} - \tilde{\boldsymbol{C}}_{kk',jj})^2. \end{aligned}$$

Since the matrix trace is invariant to orthogonal transformations, we have that $(\tilde{\boldsymbol{C}}_{kk,ii} + \tilde{\boldsymbol{C}}_{kk,jj})^2 = (\tilde{\boldsymbol{M}}_{kk,ii} + \tilde{\boldsymbol{M}}_{kk,jj})^2$, where $\tilde{\boldsymbol{M}}_{k\kappa,ij}$ is defined analogously to $\tilde{\boldsymbol{C}}_{k\kappa,ij}$. Thus we are left to optimise

$$
\begin{aligned}
(\tilde{\boldsymbol{C}}_{kk,ii} - \tilde{\boldsymbol{C}}_{kk,jj})^2 &+ \sum_{k' \neq k}^{K} (\tilde{\boldsymbol{C}}_{k'k,ii} + \tilde{\boldsymbol{C}}_{kk',ii})^2 + \sum_{k' \neq k}^{K} (\tilde{\boldsymbol{C}}_{k'k,ii} - \tilde{\boldsymbol{C}}_{kk',ii})^2 \\
&+ \sum_{k' \neq k}^{K} (\tilde{\boldsymbol{C}}_{k'k,jj} + \tilde{\boldsymbol{C}}_{kk',jj})^2 + \sum_{k' \neq k}^{K} (\tilde{\boldsymbol{C}}_{k'k,jj} - \tilde{\boldsymbol{C}}_{kk',jj})^2.
\end{aligned}
$$

Now, let $c \coloneqq \cos(\theta)$, $s \coloneqq \sin(\theta)$, and $\boldsymbol{z} \coloneqq (c, s)^\top$. Then in terms of the elements of $\tilde{\boldsymbol{\mathcal{M}}}^{(r)}$,

$$
\begin{aligned}
\tilde{\boldsymbol{C}}_{kk,ii} - \tilde{\boldsymbol{C}}_{kk,jj} &= \boldsymbol{z}^\top \begin{pmatrix} \tilde{\boldsymbol{M}}_{kk,ii} - \tilde{\boldsymbol{M}}_{kk,jj} & \tilde{\boldsymbol{M}}_{kk,ij} + \tilde{\boldsymbol{M}}_{kk,ji} \\ \tilde{\boldsymbol{M}}_{kk,ij} + \tilde{\boldsymbol{M}}_{kk,ji} & \tilde{\boldsymbol{M}}_{kk,jj} - \tilde{\boldsymbol{M}}_{kk,ii} \end{pmatrix} \boldsymbol{z} \\
&=: \boldsymbol{z}^\top \boldsymbol{\Psi}_1 \boldsymbol{z},
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k' \neq k}^{K} (\tilde{\boldsymbol{C}}_{k'k,ii} + \tilde{\boldsymbol{C}}_{kk',ii})^2 &= \boldsymbol{z}^\top \left( \sum_{k' \neq k}^{K} \begin{pmatrix} \tilde{\boldsymbol{M}}_{k'k,ii} + \tilde{\boldsymbol{M}}_{kk',ii} \\ \tilde{\boldsymbol{M}}_{k'k,ij} + \tilde{\boldsymbol{M}}_{kk',ji} \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{M}}_{k'k,ii} + \tilde{\boldsymbol{M}}_{kk',ii} \\ \tilde{\boldsymbol{M}}_{k'k,ij} + \tilde{\boldsymbol{M}}_{kk',ji} \end{pmatrix}^\top \right) \boldsymbol{z} \\
&=: \boldsymbol{z}^\top \boldsymbol{\Psi}_2 \boldsymbol{z},
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k' \neq k}^{K} (\tilde{\boldsymbol{C}}_{k'k,ii} - \tilde{\boldsymbol{C}}_{kk',ii})^2 &= \boldsymbol{z}^\top \left( \sum_{k' \neq k}^{K} \begin{pmatrix} \tilde{\boldsymbol{M}}_{k'k,ii} - \tilde{\boldsymbol{M}}_{kk',ii} \\ \tilde{\boldsymbol{M}}_{k'k,ij} - \tilde{\boldsymbol{M}}_{kk',ji} \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{M}}_{k'k,ii} - \tilde{\boldsymbol{M}}_{kk',ii} \\ \tilde{\boldsymbol{M}}_{k'k,ij} - \tilde{\boldsymbol{M}}_{kk',ji} \end{pmatrix}^\top \right) \boldsymbol{z} \\
&=: \boldsymbol{z}^\top \boldsymbol{\Psi}_3 \boldsymbol{z},
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k' \neq k}^{K} (\tilde{\boldsymbol{C}}_{k'k,jj} + \tilde{\boldsymbol{C}}_{kk',jj})^2 &= \boldsymbol{z}^\top \left( \sum_{k' \neq k}^{K} \begin{pmatrix} \tilde{\boldsymbol{M}}_{k'k,jj} + \tilde{\boldsymbol{M}}_{kk',jj} \\ -(\tilde{\boldsymbol{M}}_{k'k,ji} + \tilde{\boldsymbol{M}}_{kk',ij}) \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{M}}_{k'k,jj} + \tilde{\boldsymbol{M}}_{kk',jj} \\ -(\tilde{\boldsymbol{M}}_{k'k,ji} + \tilde{\boldsymbol{M}}_{kk',ij}) \end{pmatrix}^\top \right) \boldsymbol{z} \\
&=: \boldsymbol{z}^\top \boldsymbol{\Psi}_4 \boldsymbol{z},
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{k' \neq k}^{K} (\tilde{\boldsymbol{C}}_{k'k,jj} - \tilde{\boldsymbol{C}}_{kk',jj})^2 &= \boldsymbol{z}^\top \left( \sum_{k' \neq k}^{K} \begin{pmatrix} \tilde{\boldsymbol{M}}_{k'k,jj} - \tilde{\boldsymbol{M}}_{kk',jj} \\ -(\tilde{\boldsymbol{M}}_{k'k,jj} - \tilde{\boldsymbol{M}}_{kk',jj}) \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{M}}_{k'k,jj} + \tilde{\boldsymbol{M}}_{kk',jj} \\ -(\tilde{\boldsymbol{M}}_{k'k,ji} + \tilde{\boldsymbol{M}}_{kk',ij}) \end{pmatrix}^\top \right) \boldsymbol{z} \\
&=: \boldsymbol{z}^\top \boldsymbol{\Psi}_5 \boldsymbol{z}.
\end{aligned}
$$

Thus, minimisation of (3) is equivalent to solving

$$\max_{\boldsymbol{z}\in\mathbb{R}^2:\boldsymbol{z}^\top\boldsymbol{z}=1} \sum_{m=1}^{M} (\boldsymbol{z}^\top \boldsymbol{\Psi}_1^{(m)} \boldsymbol{z})^2 + \boldsymbol{z}^\top(\boldsymbol{\Psi}_2^{(m)} + \ldots + \boldsymbol{\Psi}_5^{(m)})\boldsymbol{z}. \tag{5}$$

The exact solution to (5) can be obtained by solving a constrained quartic equation. However, we may alternatively obtain an approximate solution to this by solving

$$\max_{\boldsymbol{z}\in\mathbb{R}^2:\boldsymbol{z}^\top\boldsymbol{z}=1} \boldsymbol{z}^\top \left( \sum_{m=1}^{M} \boldsymbol{\Psi}_1^{(m)} + \ldots + \boldsymbol{\Psi}_5^{(m)} \right) \boldsymbol{z}. \tag{6}$$

The solution to (6) can be readily obtained using a standard result in linear algebra, whereby the maximiser of this quadratic form subject to the norm constraint is given by (normalised) eigenvector of the innermost matrix in (6), associated with the largest eigenvalue (see, for example Horn and Johnson (2012) Corollary 4.3.39).

### 2.4 Distinct diagonal elements

Notice that any set of matrices $\boldsymbol{V}_1^*, \ldots, \boldsymbol{V}_K^*$ that are joint asymmetric diagonalizers of a collection of matrices $\mathscr{M}^{(1)}, \ldots, \mathscr{M}^{(M)}$ must also satisfy

$$\boldsymbol{V}_k^{*-1} \boldsymbol{M}_{kk}^{(m)} \boldsymbol{V}_k^* = \boldsymbol{D}_{kk}^{(m)} \text{ for } k = 1, \ldots, K \text{ and } m = 1, \ldots, M, \tag{7}$$

that is, any solution to (1) must also solve an associated joint diagonalization problem. In the case in which the diagonal elements of $\boldsymbol{D}_{kk}^{(m)}$ are distinct, then any solution to

$$\boldsymbol{V}_k^{**-1} \boldsymbol{M}_{kk}^{(m)} \boldsymbol{V}_k^{**} = \boldsymbol{D}_{kk}^{(m)} \tag{8}$$

must satisfy $\boldsymbol{V}_k^{**} = \boldsymbol{V}_k^* \boldsymbol{\Omega}_k \boldsymbol{\Delta}_k$ for some $N \times N$ diagonal matrix $\boldsymbol{\Omega}_k$, and $N \times N$ permutation matrix $\boldsymbol{\Delta}_k$; see, for example, Theorem 1.3.27 in Horn and Johnson (2012). If for each $k$ there is some $\boldsymbol{D}_{kk}^{(m)}$ such that the elements on the diagonal of this matrix are distinct, then there exists a block diagonal permutation matrix $\boldsymbol{\Delta}$ which contains on its diagonal the permutation matrices $\boldsymbol{\Delta}_1, \ldots, \boldsymbol{\Delta}_K$ such that

$$\boldsymbol{\Delta}^{-1}(\boldsymbol{\mathcal{V}}^{**-1} \boldsymbol{\mathcal{M}}^{(m)} \boldsymbol{\mathcal{V}}^{**}) \boldsymbol{\Delta} = \boldsymbol{\mathcal{D}}^{(m)} \text{ for } m = 1, \ldots, M. \tag{9}$$

Since permutation matrices, and by extension block permutation matrices, are orthogonal, we can exploit this structure to devise a more effective algorithm for this situation. The idea is to first solve the relatively easier problem of joint diagonalization, and then to solve for the permutation matrix $\boldsymbol{\Delta}$ in the same iterative scheme. This gives rise to the following modified procedure.

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**Algorithm:** JASYD(b)

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**Result:** Approximate joint asymmetric diagonalization of a collection of matrices.
Set $r = 0$, $\boldsymbol{\mathcal{M}}^{(m,0)} = \boldsymbol{\mathcal{M}}^{(m)}$ for $m = 1, \ldots, M$ and $\boldsymbol{\mathcal{V}}^{(0)} = \boldsymbol{I}_{NK}$.
**for** $r = 1, \ldots, R$ **do**
- $\theta_r^{**} := \arg\min f_J(\boldsymbol{\mathcal{M}}^{(1,r-1)}, \ldots, \boldsymbol{\mathcal{M}}^{(M,r-1)}; \theta)$
- $\tilde{\boldsymbol{\mathcal{V}}}^{(r)} := \boldsymbol{\mathcal{V}}^{(r-1)} \boldsymbol{G}(\theta_r^{**})$
- **for** $m = 1, \ldots, M$ **do**
  - $\tilde{\boldsymbol{\mathcal{M}}}^{(m,r)} := (\tilde{\boldsymbol{\mathcal{V}}}^{(r)})^{-1} \boldsymbol{\mathcal{M}}^{(m,r-1)} \tilde{\boldsymbol{\mathcal{V}}}^{(r)}$
- **end**
- $\phi_r^* := \arg\min f_H(\tilde{\boldsymbol{\mathcal{M}}}^{(1,r)}, \ldots, \tilde{\boldsymbol{\mathcal{M}}}^{(M,r)}; \phi)$
- $\tilde{\tilde{\boldsymbol{\mathcal{V}}}}^{(r)} := \tilde{\boldsymbol{\mathcal{V}}}^{(r)} \boldsymbol{H}(\phi_r^*)$
- **for** $m = 1, \ldots, M$ **do**
  - $\tilde{\tilde{\boldsymbol{\mathcal{M}}}}^{(m,r)} := (\tilde{\tilde{\boldsymbol{\mathcal{V}}}}^{(r)})^{-1} \tilde{\boldsymbol{\mathcal{M}}}^{(m,r)} \tilde{\tilde{\boldsymbol{\mathcal{V}}}}^{(r)}$
- **end**
- $\theta_r^* := \arg\min f_G(\tilde{\tilde{\boldsymbol{\mathcal{M}}}}^{(1,r)}, \ldots, \tilde{\tilde{\boldsymbol{\mathcal{M}}}}^{(M,r)}; \theta)$
- $\boldsymbol{\mathcal{V}}^{(r)} := \tilde{\tilde{\boldsymbol{\mathcal{V}}}}^{(r)} \boldsymbol{G}(\theta_r^*)$
- **for** $m = 1, \ldots, M$ **do**
  - $\boldsymbol{\mathcal{M}}^{(m,r)} := (\boldsymbol{\mathcal{V}}^{(r)})^{-1} \tilde{\tilde{\boldsymbol{\mathcal{M}}}}^{(m,r)} \boldsymbol{\mathcal{V}}^{(r)}$
- **end**

**end**

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This modified scheme employs an additional optimisation step which computes the parameter $\theta^{**}$ that optimises the following criterion:

$$f_J(\boldsymbol{\mathcal{M}}^{(1,r)}, \ldots, \boldsymbol{\mathcal{M}}^{(M,r)}; \theta) = \sum_{m=1}^{M} ||\text{off}(\boldsymbol{G}^\top(\theta)\text{blkdiag}(\boldsymbol{\mathcal{M}}^{(m,r)})\boldsymbol{G}(\theta))||_F^2. \tag{10}$$

To find a minimiser of this function we can follow the same steps as used to obtain (9), simply setting the elements outside of the block diagonal to zero. Doing this we find optimising (10) to be equivalent to

$$\max_{\boldsymbol{z} \in \mathbb{R}^2 : \boldsymbol{z}^\top \boldsymbol{z} = 1} \sum_{m=1}^{M} (\boldsymbol{z}^\top \boldsymbol{\Psi}_1^{(m)} \boldsymbol{z})^2.$$

Noticing that

$$\boldsymbol{z}^\top \boldsymbol{\Psi}_1^{(m)} \boldsymbol{z} = \begin{pmatrix} c^2 - s^2 & 2cs \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{M}}_{kk,ii}^{(m)} - \tilde{\boldsymbol{M}}_{kk,jj}^{(m)} \\ \tilde{\boldsymbol{M}}_{kk,ij}^{(m)} + \tilde{\boldsymbol{M}}_{kk,ji}^{(m)} \end{pmatrix} =: \boldsymbol{w}^\top \boldsymbol{\phi}^{(m)},$$

and also that $||\boldsymbol{w}||_2 = 1$, we can alternatively solve

$$\max_{\boldsymbol{w} \in \mathbb{R}^2 : \boldsymbol{w}^\top \boldsymbol{w} = 1} \sum_{m=1}^{M} \boldsymbol{w}^\top \boldsymbol{\phi}^{(m)} \boldsymbol{\phi}^{(m)\top} \boldsymbol{w}. \tag{11}$$

As with (6), the solution to (11), denoted $w^*$, is easy to obtain, with which the optimal angle can be computed by

$$\begin{aligned} \frac{1}{2} \tan^{-1} \left( \frac{w_2^*}{w_1^*} \right) &= \frac{1}{2} \tan^{-1} \left( \frac{2 \cos(\theta^*) \sin(\theta^*)}{\cos^2(\theta^*) - \sin^2(\theta^*)} \right) \\ &= \frac{1}{2} \tan^{-1} \left( \frac{\sin(2\theta^*)}{\cos(2\theta^*)} \right) = \theta^{**}. \end{aligned}$$

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